

# LEFSCHETZ AND HIRZEBRUCH-RIEMANN-ROCH FORMULAS VIA NONCOMMUTATIVE MOTIVES

DENIS-CHARLES CISINSKI AND GONALO TABUADA

ABSTRACT. V. Lunts has recently established Lefschetz fixed point theorems for Fourier-Mukai functors and dg algebras. In the same vein, D. Shklyarov introduced the noncommutative analogue of the Hirzebruch-Riemann-Roch theorem. In this note, making use of the theory of noncommutative motives, we show how these beautiful theorems can be understood as instantiations of more general results.

## 1. STATEMENT OF RESULTS

Let  $k$  be a field. Lunts' results can be resumed as follows:

**Theorem 1.1.** (see [9, Thms 1.1, 1.2 and 1.4]) *Let  $X$  be a smooth projective  $k$ -variety and  $E$  a bounded complex of coherent sheaves on  $X \times X$ . Then, the following equality holds*

$$(1.2) \quad \sum_i (-1)^i \dim HH_i(E) = \sum_j (-1)^j \operatorname{Tr} HH_j(\Phi_E),$$

where  $\Phi_E$  is the associated Fourier-Mukai functor. Moreover, when  $k = \mathbb{C}$ , we have

$$(1.3) \quad \sum_i (-1)^i \dim HH_i(E) = \operatorname{Tr} H^{\text{even}}(\Phi_E) - \operatorname{Tr} H^{\text{odd}}(\Phi_E),$$

with  $H^{\text{even}}$  and  $H^{\text{odd}}$  the even and odd parts of singular cohomology.

Let  $A$  be a smooth and proper dg  $k$ -algebra and  $M$  a perfect complex of  $A$ -bimodules. Then, the following equality holds

$$(1.4) \quad \sum_i (-1)^i \dim HH_i(A; M) = \sum_j (-1)^j \operatorname{Tr} HH_j(\Phi_M).$$

Shklyarov's result can be stated as follows.

**Theorem 1.5.** (see [14, Thm. 3]) *Let  $A$  be a proper dg  $k$ -algebra and  $M$  and  $N$  perfect complexes of  $A$ -modules. Then, the following equality holds*

$$(1.6) \quad \chi(M, N) = \langle eu(N), eu(DM) \rangle,$$

where  $DM$  is the  $k$ -linear dual of  $M$  and  $eu$  the Euler character.

In this note we prove the following general motivic results (over a base ring  $k$ ) and show how equalities (1.2)-(1.4) and (1.6) can be recovered from them.

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**Theorem 1.7** (Noncommutative Lefschetz). *Let  $\mathcal{A}$  be a smooth and proper dg category (see §2),  $M$  a perfect  $\mathcal{A}$ -bimodule, and  $L : \mathbf{dgc} \rightarrow (\mathbf{D}, \otimes, \mathbf{1})$  an additive invariant (see Def. 4.1) which is symmetric monoidal when restricted to smooth and proper dg categories. Assume that the base ring  $k$  is local (or more generally that  $K_0(k) = \mathbb{Z}$ ). Then, the following equality in  $\mathrm{End}_{\mathbf{D}}(\mathbf{1})$  holds*

$$(1.8) \quad \sum_i (-1)^i \mathrm{rk} H_i(\mathcal{A}; M) = \mathrm{Tr} L(\Phi_M).$$

**Theorem 1.9** (Noncommutative Hirzebruch-Riemann-Roch). *Let  $\mathcal{A}$  be a proper dg category,  $M$  and  $N$  perfect  $\mathcal{A}$ -modules, and  $L : \mathbf{dgc} \rightarrow \mathbf{D}$  an additive invariant which is symmetric monoidal when restricted to proper dg categories. Then, the following equality in  $\mathrm{End}_{\mathbf{D}}(\mathbf{1})$  holds*

$$(1.10) \quad \mathrm{ch}_L(\widehat{\mathcal{A}}(M, N)) = \langle \mathrm{ch}_L(N), \mathrm{ch}_L(DM) \rangle,$$

where  $\widehat{\mathcal{A}}(M, N)$  is the complex of morphisms from  $M$  and  $N$  (see §2) and  $\mathrm{ch}_L$  the Chern character associated to  $L$ ; see §6. Moreover, when  $k$  is local (or more generally when  $K_0(k) = \mathbb{Z}$ ) and  $\mathcal{A}$  is smooth, equality (1.10) follows from equality (1.8).

## 2. DIFFERENTIAL GRADED CATEGORIES

Let  $k$  be a (fixed) base commutative ring. A *differential graded (=dg) category*  $\mathcal{A}$  is a category enriched over complexes of  $k$ -modules (morphism sets are complexes) in such a way that composition fulfills the Leibniz rule:  $d(f \circ g) = d(f) \circ g + (-1)^{\deg(f)} f \circ d(g)$ ; consult Keller's ICM address [5]. Let us denote by  $\mathbf{dgc}$  the category of (small) dg categories. Given any (dg)  $k$ -algebra  $A$ , we will write  $\underline{A}$  for the dg category with a single object and with  $A$  as the (dg)  $k$ -algebra of endomorphisms. Given a dg category  $\mathcal{A}$ , we will write  $\widehat{\mathcal{A}}$  for the associated dg category of (right)  $\mathcal{A}$ -modules. A dg functor  $\mathcal{A} \rightarrow \mathcal{B}$  is called a *Morita equivalence* if the induced extension-of-scalars functor  $\mathcal{D}(\mathcal{A}) \xrightarrow{\sim} \mathcal{D}(\mathcal{B})$  on derived categories is an equivalence; see [5, §4.6]. The tensor product of  $k$ -algebras extends naturally to dg categories, giving rise to a symmetric monoidal structure  $- \otimes -$  on  $\mathbf{dgc}$ . The  $\otimes$ -unit is the dg category  $\underline{k}$ . Finally, following Kontsevich [7, 8], a dg category  $\mathcal{A}$  is called *smooth* if it is perfect as a bimodule over itself, and *proper* if for every ordered pair of objects  $(x, y)$  the complex of  $k$ -modules  $\mathcal{A}(x, y)$  is perfect.

## 3. NONCOMMUTATIVE CHOW MOTIVES

Let  $\mathbf{Hmo}$  be the localization of  $\mathbf{dgc}$  with respect to the class of Morita equivalences. The tensor product of dg categories can be naturally derived  $- \otimes^{\mathbb{L}} -$ , giving thus rise to a symmetric monoidal structure on  $\mathbf{Hmo}$ ; see [15, Remark 5.11]. Given any two dg categories  $\mathcal{A}$  and  $\mathcal{B}$ , there is a bijection

$$(3.1) \quad \mathrm{Iso} \, \mathrm{rep}(\mathcal{A}, \mathcal{B}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Hmo}}(\mathcal{A}, \mathcal{B}) \quad M \mapsto \Phi_M,$$

where  $\mathrm{Iso}$  denotes the set of isomorphism classes,  $\Phi_M = - \otimes_{\mathcal{A}}^{\mathbb{L}} M$ , and  $\mathrm{rep}(\mathcal{A}, \mathcal{B})$  is the full triangulated subcategory of  $\mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes^{\mathbb{L}} \mathcal{B})$  spanned by the  $\mathcal{A}$ - $\mathcal{B}$ -bimodules  $M$  such that for every object  $x \in \mathcal{A}$  the associated  $\mathcal{B}$ -module  $M(-, x)$  is compact in  $\mathcal{D}(\mathcal{B})$ ; consult [15, Cor. 5.10] for further details. Moreover, under the above bijection (3.1), the composition law in  $\mathbf{Hmo}$  corresponds to the (derived) tensor product of bimodules.

Now, let  $\mathbf{Hmo}_0$  be the additive category with the same objects as  $\mathbf{Hmo}$  and with morphisms given by  $\mathbf{Hom}_{\mathbf{Hmo}_0}(\mathcal{A}, \mathcal{B}) := K_0\mathrm{rep}(\mathcal{A}, \mathcal{B})$ , where  $K_0\mathrm{rep}(\mathcal{A}, \mathcal{B})$  stands for the Grothendieck group of the triangulated category  $\mathrm{rep}(\mathcal{A}, \mathcal{B})$ . The composition law is the induced one; consult [15, §6]. The derived tensor product on  $\mathbf{Hmo}$  extends by linearity to  $\mathbf{Hmo}_0$  giving rise to a symmetric monoidal structure. Moreover, there is a natural sequence of symmetric monoidal functors

$$(3.2) \quad \mathcal{U} : \mathrm{dgc} \longrightarrow \mathbf{Hmo} \longrightarrow \mathbf{Hmo}_0.$$

Finally, let  $\mathbf{Hmo}_0^{\mathrm{sp}}$  be the full subcategory of  $\mathbf{Hmo}_0$  formed by the smooth and proper dg categories. When  $\mathcal{A}$  is smooth and proper, we have an equivalence  $\mathrm{rep}(\mathcal{A}, \mathcal{B}) \simeq \mathcal{D}_c(\mathcal{A}^{\mathrm{op}} \otimes^{\mathbb{L}} \mathcal{B})$  of triangulated categories and so we obtain the following description  $\mathbf{Hom}_{\mathbf{Hmo}_0^{\mathrm{sp}}}(\mathcal{A}, \mathcal{B}) = K_0(\mathcal{D}_c(\mathcal{A}^{\mathrm{op}} \otimes^{\mathbb{L}} \mathcal{B})) = K_0(\mathcal{A}^{\mathrm{op}} \otimes^{\mathbb{L}} \mathcal{B})$  of the morphism sets in  $\mathbf{Hmo}_0^{\mathrm{sp}}$ . The category  $\mathbf{NChow}$  of *noncommutative Chow motives* is by definition the pseudo-abelian envelope of  $\mathbf{Hmo}_0^{\mathrm{sp}}$ . Note that since smooth and proper dg categories are stable under (derived) tensor product, the category  $\mathbf{NChow}$  is symmetric monoidal.

#### 4. SYMMETRIC MONOIDAL ADDITIVE INVARIANTS

Let  $\mathcal{A}$  be a dg category. Consider the dg category  $T(\mathcal{A})$  whose objects are the pairs  $(i, x)$ , with  $i \in \{1, 2\}$  and  $x$  an object of  $\mathcal{A}$ . The complex of morphisms  $T(\mathcal{A})((i, x), (i', x'))$  equals  $\mathcal{A}(x, x')$  if  $i' \geq i$  and is 0 otherwise. Composition is induced from  $\mathcal{A}$ ; consult [15, §4] for details. Note that we have two natural inclusion dg functors

$$\iota_1 : \mathcal{A} \longrightarrow T(\mathcal{A}) \quad \iota_2 : \mathcal{A} \longrightarrow T(\mathcal{A}).$$

*Definition 4.1.* Let  $L : \mathrm{dgc} \rightarrow \mathbf{D}$  be a functor with values in an idempotent complete additive category  $\mathbf{D}$ . We say that  $L$  is an *additive invariant* if it satisfies the following two conditions:

- (i) it sends Morita equivalences to isomorphisms;
- (ii) the above inclusion dg functors  $\iota_1$  and  $\iota_2$  induce an isomorphism<sup>1</sup>

$$[L(\iota_1) \ L(\iota_2)] : L(\mathcal{A}) \oplus L(\mathcal{A}) \xrightarrow{\sim} L(T(\mathcal{A})).$$

Let us now describe some examples of additive invariants which are moreover symmetric monoidal.

*Example 4.2* (Hochschild homology). Let  $\mathcal{D}(k)$  be the derived category of the base commutative ring  $k$ . By construction, this category is idempotent complete, additive, and symmetric monoidal. As explained in [2, Example 7.9], the Hochschild homology functor

$$(4.3) \quad HH : \mathrm{dgc} \longrightarrow \mathcal{D}(k)$$

sends Morita equivalences to isomorphisms and is symmetric monoidal. Moreover, since localization implies condition (ii) of Definition 4.1, we conclude from *loc. cit.* that (4.3) is a symmetric monoidal additive invariant.

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<sup>1</sup>Condition (ii) can be equivalently formulated in terms of a general semi-orthogonal decomposition in the sense of Bondal-Orlov; see [15, Thm. 6.3(4)].

*Example 4.4* (Mixed complexes). Following Kassel [4, §1], a *mixed complex*  $(N, b, B)$  is a  $\mathbb{Z}$ -graded  $k$ -module  $\{N_n\}_{n \in \mathbb{Z}}$  endowed with a degree  $+1$  endomorphism  $b$  and a degree  $-1$  endomorphism  $B$  satisfying the relations  $b^2 = B^2 = Bb + bB = 0$ . Equivalently, a mixed complex is a right dg module over the dg algebra  $\Lambda := k[\epsilon]/\epsilon^2$ , where  $\epsilon$  is of degree  $-1$  and  $d(\epsilon) = 0$ . Let  $\mathcal{D}(\Lambda)$  be the derived category of mixed complexes. By construction, this category is idempotent complete and additive. Moreover, it is symmetric monoidal with the tensor product defined on the underlying complexes. As explained in [2, Example 7.10], the mixed complex functor

$$(4.5) \quad C : \text{dgc}at \longrightarrow \mathcal{D}(\Lambda)$$

sends Morita equivalences to isomorphisms and is symmetric monoidal. Since localization implies condition (ii) of Definition 4.1, we then conclude from *loc. cit.* that (4.5) is a symmetric monoidal additive invariant.

*Example 4.6* (Periodic cyclic homology). As explained by Kassel in [4, page 210], there is a *2-perioditization functor* sending a mixed complex  $(N, b, B)$  to the  $\mathbb{Z}/2$ -graded complex

$$\prod_{n \text{ even}} N_n \xrightleftharpoons[b+B]{b+B} \prod_{n \text{ odd}} N_n.$$

This functor preserves weak equivalences and when combined with  $C$  gives rise to periodic cyclic homology

$$(4.7) \quad HP : \text{dgc}at \xrightarrow{C} \mathcal{D}(\Lambda) \longrightarrow \mathcal{D}_{\mathbb{Z}/2}(k).$$

Here,  $\mathcal{D}_{\mathbb{Z}/2}(k)$  stands for the derived category of  $\mathbb{Z}/2$ -graded complexes, which by construction is idempotent complete, additive and symmetric monoidal. Since the 2-perioditization functor is additive and  $C$  is an additive invariant, we conclude that the functor (4.7) is also an additive invariant. The 2-perioditization functor is *not* symmetric monoidal since it uses infinite products and these do not commute with the tensor product. Nevertheless, when  $k$  is a field, the functor (4.7) is symmetric monoidal when restricted to smooth and proper dg categories; see [13, Thm. 7.2].

## 5. NONCOMMUTATIVE LEFSCHETZ

**Proof of Theorem 1.7.** As proved in [15], (3.2) is the *universal additive invariant*. Hence,  $L$  gives rise to a (unique) additive functor  $\overline{L} : \mathbf{Hmo}_0 \rightarrow \mathbf{D}$  such that  $\overline{L} \circ \mathcal{U} = L$ . By hypothesis,  $L$  (and hence  $\overline{L}$ ) is symmetric monoidal when restricted to smooth and proper dg categories. Therefore, since  $\mathbf{D}$  is idempotent complete, we obtain a well-defined symmetric monoidal additive functor

$$(5.1) \quad \overline{L} : \mathbf{NChow} \longrightarrow (\mathbf{D}, \otimes, \mathbf{1}).$$

As proved in [2, Thm. 4.8], the rigid objects in the symmetric monoidal category  $\mathbf{Hmo}$  are precisely the smooth and proper dg categories. As a consequence, the category  $\mathbf{NChow}$  is rigid. We can then, in particular, compute the categorical trace of any of its endomorphisms and hence the categorical trace of any endomorphism in the essential image of the functor (5.1). Since  $\mathcal{A}$  is smooth and proper and  $M$  is a perfect  $\mathcal{A}$ -bimodule (i.e.  $M \in \mathcal{D}_c(\mathcal{A}^{\text{op}} \otimes^{\mathbb{L}} \mathcal{A})$ ), we have an associated endomorphism

$\Phi_{[M]} : \mathcal{A} \rightarrow \mathcal{A}$  of  $\mathcal{A}$  in  $\text{NChow}$ . The functor (5.1) is symmetric monoidal and so we conclude that the image of  $\text{Tr}(\Phi_{[M]})$  under the induced homomorphism

$$K_0(\mathcal{D}_c(k)) \simeq K_0(\underline{k}^{\text{op}} \otimes^{\mathbb{L}} \underline{k}) \simeq \text{End}_{\text{NChow}}(\underline{k}) \longrightarrow \text{End}_{\mathbb{D}}(\mathbf{1})$$

agrees with the categorical trace of the endomorphism  $\overline{L}(\Phi_{[M]})$  of  $L(\mathcal{A})$ . As proved in [11, Prop. 4.3], the following equality holds

$$(5.2) \quad \text{Tr}(\Phi_{[M]}) = [HH(\mathcal{A}; M)] \in K_0(\mathcal{D}_c(k)).$$

By hypothesis,  $k$  is a local ring and so we have a natural identification

$$(5.3) \quad K_0(\mathcal{D}_c(k)) \simeq K_0(k) \xrightarrow{\sim} \mathbb{Z} \quad N \mapsto \sum_i (-1)^i \text{rk} H_i(N),$$

where  $\text{rk} H_i(N)$  stands for the rank of the  $i^{\text{th}}$ -homology group of a perfect complex  $N$  of  $k$ -modules. By combining (5.2) with (5.3) we then obtain the following equality

$$\text{Tr}(\Phi_{[M]}) = \sum_i (-1)^i \text{rk} HH_i(\mathcal{A}; M).$$

Finally, since  $\overline{L}(\Phi_{[M]}) = L(\Phi_M)$ , we conclude that both terms of (1.8) agree.

**Recovering equality (1.4) from equality (1.8).** When  $k$  is a field,  $L$  is the Hochschild homology functor described in Example 4.2, and  $\mathcal{A}$  is the dg category  $\underline{A}$  associated to a smooth and proper dg algebra  $A$ , equality (1.8) clearly reduces to Lunts' equality (1.4); note that  $\text{End}_{\mathcal{D}(k)}(k) \simeq k$  and that the categorical trace in  $\mathcal{D}_c(k)$  of the endomorphism  $HH(\Phi_M)$  identifies with  $\sum_j (-1)^j \text{Tr} HH_j(\Phi_M)$ .

**Recovering equalities (1.2)-(1.3) from equality (1.8).** In this subsection  $k$  will be a field. Given a quasi-compact and separated  $k$ -scheme  $X$ , it is well-known that its derived category  $\mathcal{D}_{\text{perf}}(X)$  of perfect complexes of  $\mathcal{O}_X$ -modules admits a differential graded (=dg) enhancement  $\mathcal{D}_{\text{perf}}^{\text{dg}}(X)$ ; consult Lunts-Orlov [10]. Moreover, when  $X$  is smooth and proper the associated dg category  $\mathcal{D}_{\text{perf}}^{\text{dg}}(X)$  is smooth and proper in the sense of Kontsevich; see for instance [2, Example 4.5(i)]. Finally notice that since  $X$  is regular, every bounded complex of coherent sheaves on  $X$  is perfect (up to quasi-isomorphism).

Now, let  $X$  and  $Y$  be two smooth projective  $k$ -varieties. As proved by Toën in [17, §8.3], the bijection (3.1) (with  $\mathcal{A} = \mathcal{D}_{\text{perf}}^{\text{dg}}(X)$  and  $\mathcal{B} = \mathcal{D}_{\text{perf}}^{\text{dg}}(Y)$ ) corresponds to

$$(5.4) \quad \text{Iso} \mathcal{D}_{\text{perf}}(X \times Y) \xrightarrow{\sim} \text{Hom}_{\text{Hmo}}(\mathcal{D}_{\text{perf}}^{\text{dg}}(X), \mathcal{D}_{\text{perf}}^{\text{dg}}(Y)) \quad E \mapsto \Phi_E,$$

where  $\Phi_E$  is the *Fourier-Mukai dg functor*

$$\Phi_E : \mathcal{D}_{\text{perf}}^{\text{dg}}(X) \longrightarrow \mathcal{D}_{\text{perf}}^{\text{dg}}(Y) \quad \mathcal{F} \mapsto \mathbb{R}(\pi_Y)_*(\mathbb{L}(\pi_X)^*(\mathcal{F}) \otimes^{\mathbb{L}} E).$$

**Corollary 5.5.** *Let  $X$  be a smooth projective  $k$ -variety and  $E \in \mathcal{D}_{\text{perf}}(X \times X)$ .*

- (i) *By taking  $\mathcal{A} = \mathcal{D}_{\text{perf}}^{\text{dg}}(X)$ ,  $M = E$  and  $L = HH$  in equality (1.8), we recover Lunts' equality (1.2).*
- (ii) *By taking  $\mathcal{A} = \mathcal{D}_{\text{perf}}^{\text{dg}}(X)$ ,  $M = E$  and  $L = HP$  in equality (1.8), and by furthermore assuming  $k = \mathbb{C}$ , we recover Lunts' equality (1.3).*

*Proof.* Let us start with item (i). As proved by Keller [6], the Hochschild homology (with coefficients) of the dg category  $\mathcal{D}_{\text{perf}}^{\text{dg}}(X)$  agrees with the Hochschild homology of  $X$  in the sense of Weibel [18]. Moreover, as explained in [1, §4.2], Weibel's

definition is equivalent to Caldararu-Willerton's definition of Hochschild homology. As a consequence, since  $k$  is a field, we obtain the following equality

$$(5.6) \quad \sum_i (-1)^i \mathrm{rk} HH_i(\mathcal{D}_{\mathrm{perf}}^{\mathrm{dg}}(\mathcal{A}); E) = \sum_i (-1)^i \dim HH_i(E),$$

where  $HH_i(E)$  is as in [9, Def. 3.7]. The proof now follows from the fact that the categorical trace in  $\mathcal{D}_c(k)$  of  $HH(\Phi_E)$  identifies with  $\sum_j (-1)^j \mathrm{Tr} HH_j(\Phi_E)$ .

Let us now prove item (ii). As proved by Keller [6], the periodic cyclic homology of the dg category  $\mathcal{D}_{\mathrm{perf}}^{\mathrm{dg}}(X)$  agrees with the periodic cyclic homology of  $X$  in the sense of Weibel [18]. Since  $k$  is a field of characteristic zero and by hypothesis  $X$  is smooth, the Hochschild-Konstant-Rosenberg theorem [18] furnish us the following isomorphism

$$HP(\mathcal{D}_{\mathrm{perf}}^{\mathrm{dg}}(X)) \simeq \left( \bigoplus_{n \text{ even}} H_{dR}^n(X), \bigoplus_{n \text{ odd}} H_{dR}^n(X) \right),$$

where  $H_{dR}^*$  stands for de Rham cohomology. Furthermore, since  $k = \mathbb{C}$ , Grothendieck comparison isomorphism [3] allows us to replace de Rham cohomology with singular cohomology  $H^*$ . Hence, the right hand-side of equality (1.8) (with  $L = HP$  and  $M = E$ ) identifies with the super-trace

$$\mathrm{Tr} H^{\mathrm{even}}(\Phi_E) - \mathrm{Tr} H^{\mathrm{odd}}(\Phi_E).$$

The proof now follows from the above equality (5.6).  $\square$

## 6. NONCOMMUTATIVE HIRZEBRUCH-RIEMMAN-ROCH

Let  $\mathcal{A}$  be a proper dg category. Consider the following  $(\mathcal{A} \otimes^{\mathbb{L}} \mathcal{A}^{\mathrm{op}})\text{-}\underline{k}$ -bimodule

$$\underline{\mathcal{A}} : \mathcal{A}^{\mathrm{op}} \otimes^{\mathbb{L}} \mathcal{A} \longrightarrow \mathcal{C}_{\mathrm{dg}}(k) \quad (x, y) \mapsto \mathcal{A}(x, y),$$

where  $\mathcal{C}_{\mathrm{dg}}(k)$  is the dg category of complexes of  $k$ -modules. Since  $\mathcal{A}$  is proper,  $\underline{\mathcal{A}} \in \mathrm{rep}(\mathcal{A} \otimes^{\mathbb{L}} \mathcal{A}^{\mathrm{op}}, \underline{k})$ . As a consequence, we have a morphism  $\Phi_{\underline{\mathcal{A}}} : \mathcal{A} \otimes^{\mathbb{L}} \mathcal{A}^{\mathrm{op}} \rightarrow \underline{k}$  in  $\mathrm{Hmo}$ . Now, let  $L : \mathrm{dgcat} \rightarrow (\mathbf{D}, \otimes, \mathbf{1})$  be an additive invariant which is symmetric monoidal when restricted to proper dg categories. By applying it to  $\Phi_{\underline{\mathcal{A}}}$  we obtain then a well-defined bilinear pairing

$$(6.1) \quad \langle -, - \rangle : L(\mathcal{A}) \otimes L(\mathcal{A}^{\mathrm{op}}) \longrightarrow \mathbf{1}.$$

Moreover, as explained in the proof of Theorem 1.7,  $L$  gives rise to a (unique) additive functor  $\overline{L} : \mathrm{Hmo}_0 \rightarrow \mathbf{D}$  such that  $\overline{L} \circ \mathcal{U} = L$ . We obtain then an induced natural transformation of functors

$$(6.2) \quad K_0(-) = \mathrm{Hom}_{\mathrm{Hmo}_0}(\underline{k}, -) \Rightarrow \mathrm{Hom}_{\mathbf{D}}(\mathbf{1}, L(-)).$$

Given a perfect  $\mathcal{A}$ -module  $M$ , we will denote by  $ch_L(M)$  the image of the associated morphism  $\Phi_{[M]} : \underline{k} \rightarrow \mathcal{A}$  in  $\mathrm{Hmo}_0$  under the natural transformation (6.2). Finally, the right hand-side of equality (1.10) is given by the following composition

$$(6.3) \quad \mathbf{1} \xrightarrow{L(\Phi_{[N]}) \otimes L(\Phi_{[DM]})} L(\mathcal{A}) \otimes L(\mathcal{A}^{\mathrm{op}}) \xrightarrow{\langle -, - \rangle} \mathbf{1};$$

notice that the  $k$ -linear dual  $DM$  of  $M$  is a perfect  $\mathcal{A}^{\mathrm{op}}$ -module.

**Proof of Theorem 1.9.** By hypothesis  $L$  is symmetric monoidal when restricted to proper dg categories. Hence, the above composition (6.3) agrees with the image under  $L$  of the following composition

$$\underline{k} \xrightarrow{\Phi_N \otimes \Phi_{DM}} L(\mathcal{A}) \otimes L(\mathcal{A}^{\text{op}}) \xrightarrow{\underline{A}} \underline{k},$$

i.e. it agrees with the Chern character of the perfect  $\underline{k}$ -module  $(N \otimes^{\mathbb{L}} DM) \otimes_{\mathcal{A} \otimes \mathcal{A}^{\text{op}}}^{\mathbb{L}} \underline{A}$ . Since we have natural quasi-isomorphisms of complexes of  $k$ -modules

$$(N \otimes^{\mathbb{L}} DM) \otimes_{\mathcal{A} \otimes \mathcal{A}^{\text{op}}}^{\mathbb{L}} \mathcal{A} \simeq N \otimes_{\mathcal{A}}^{\mathbb{L}} DM \simeq \widehat{\mathcal{A}}(M, N)$$

the proof of the first claim is finished.

Let us now prove the second claim. We will write  $DM^{\#}$  for the  $\mathcal{A}$ -module associated to the  $k$ -linear dual  $DM$  of  $M$ . Since  $\mathcal{A}$  is proper we obtain then a well-defined morphism  $\Phi_{DM^{\#}} : \mathcal{A} \rightarrow \underline{k}$  in  $\text{Hmo}$ . We start by showing that the left hand-side of (1.8) (with  $M$  the  $\mathcal{A}$ -bimodule  $DM^{\#} \otimes^{\mathbb{L}} N$ ) agrees with  $ch_L(\widehat{\mathcal{A}}(M, N))$ . As explained in the proof of Theorem 1.7, the left hand-side of (1.8) agrees with the image under

$$(6.4) \quad K_0(\mathcal{D}_c(k)) \simeq \text{End}_{\text{NChow}}(\underline{k}) \longrightarrow \text{End}_{\mathbb{D}}(\mathbf{1})$$

of the categorical trace of the endomorphism  $\Phi_{[DM^{\#} \otimes^{\mathbb{L}} N]} : \mathcal{A} \rightarrow \mathcal{A}$  of  $\mathcal{A}$  in  $\text{NChow}$ . Moreover, as proved in [12, Prop. 6.1] (with  $X = M$  and  $Y = N$ ) this categorical trace agrees with  $\chi(M, N) \in K_0(\mathcal{D}_c(k)) \simeq \mathbb{Z}$ , i.e. with the Euler characteristic of the complex  $\widehat{\mathcal{A}}(M, N)$ ; recall from *loc. cit.* that every smooth and proper dg category is Morita equivalent to a smooth and proper dg algebra. Since the isomorphism  $K_0(\mathcal{D}_c(k)) \simeq \mathbb{Z}$  is given by the Euler characteristic as in (5.3), we then conclude that  $ch_L(\widehat{\mathcal{A}}(M, N))$  agrees with the image of  $\chi(M, N)$  under (6.4).

Let us now show that the right hand-side of equality (1.8) (with  $M = DM^{\#} \otimes^{\mathbb{L}} N$ ) agrees with  $\langle ch_L(N), ch_L(DM) \rangle$ . Recall from [2, Thm. 4.8] that the dual of  $\mathcal{A}$  is  $\mathcal{A}^{\text{op}}$  and that the evaluation map is given by the morphism  $\Phi_{\underline{A}} : \mathcal{A} \otimes^{\mathbb{L}} \mathcal{A}^{\text{op}} \rightarrow \underline{k}$ . Since by hypothesis  $L$  is symmetric monoidal when restricted to smooth and proper dg categories and the  $\underline{k}$ -( $\mathcal{A}^{\text{op}} \otimes^{\mathbb{L}} \mathcal{A}$ )-bimodule associated to  $DM^{\#} \otimes^{\mathbb{L}} N$  is precisely  $DM \otimes^{\mathbb{L}} N$ , we observe that the right hand-side of (1.8) agrees in this case with the image under  $L$  of the following composition

$$(6.5) \quad \underline{k} \xrightarrow{\Phi_{DM \otimes^{\mathbb{L}} N}} \mathcal{A}^{\text{op}} \otimes^{\mathbb{L}} \mathcal{A} \simeq \mathcal{A} \otimes^{\mathbb{L}} \mathcal{A}^{\text{op}} \xrightarrow{\Phi_{\underline{A}}} \underline{k}.$$

Finally, since  $\Phi_{DM \otimes^{\mathbb{L}} N} = \Phi_{DM} \otimes^{\mathbb{L}} \Phi_N$ , we conclude from composition (6.3) that the image of (6.5) under  $L$  is precisely  $\langle ch_L(N), ch_L(DM) \rangle$ .

**Recovering equality (1.6) from equality (1.10).** When  $k$  is a field,  $L$  is the Hochschild homology functor described in Example 4.2, and  $\mathcal{A}$  is the dg category  $\underline{A}$  associated to a proper dg algebra  $A$ , equality (1.10) reduces to equality (1.6). This follows from the fact that Chern character  $ch_{HH}$  agrees with the Dennis trace map (see [16, Thm. 2.8]) and from the fact that the Euler character  $eu$  is precisely the image under the Dennis trace map; consult Shklyarov's notations [14].

**Negative Chern character.** When  $L$  is the mixed complex functor described in Example 4.4 and  $\mathcal{A}$  is the dg category  $\underline{A}$  associated to a proper dg algebra  $A$ , equality (1.10) reduces to

$$ch^-(\widehat{\mathcal{A}}(M, N)) = \langle ch^-(N), ch^-(DM) \rangle,$$



where  $ch^-$  is the classical negative Chern character with values in negative cyclic homology; see [16, Thm. 2.8].

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UNIVERSITÉ PAUL SABATIER, INSTITUT DE MATHÉMATIQUES DE TOULOUSE, 118 ROUTE DE NARBONNE, F-31062 TOULOUSE CEDEX 9

*E-mail address*: [denis-charles.cisinski@math.univ-toulouse.fr](mailto:denis-charles.cisinski@math.univ-toulouse.fr)

*URL*: <http://www.math.univ-toulouse.fr/~dcisinsk/>

GONALO TABUADA, DEPARTMENT OF MATHEMATICS, MIT, CAMBRIDGE, MA 02139, USA

*E-mail address*: [tabuada@math.mit.edu](mailto:tabuada@math.mit.edu)